THE OPTIMIZATION OF MULTI-STAGE ORBIT TRANSFER PROCESSES BY DYNAMIC PROGRAMMING

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P-2177-1

January 24, 1961

Pages bearing P-2177-1 have been revised
8/30/61

This paper has been accepted for publication by the Journal of The American Rocket Society.

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INTRODUCTION

The purpose of this paper is to expand the idea suggested in Ref. 1 of using a set of variation of parameter equations in an orbit transfer process. The orbit transfer process in the case considered here involves the correction of the motion of a space vehicle in some optimum manner.

Reference 1 considers two orbit transfer methods. One is a two-stage exact** method. The other is a single-stage approximate method which minimizes the sum of the weighted squares of the errors in the orbital parameters existing at the termination of thrust. A generalization of the latter method is the one discussed here.

Other references considering optimization of orbit transfer processes are included in the list of references.[2,3,4,5] These papers use conventional minimization techniques, i.e., setting partial derivatives of functions equal to zero, trial and error methods, etc. This paper considers the optimization of multi-stage orbit transfer processes by the method of dynamic programming. Further, due to the linearity of the state transformation equations and the quadratic system performance index used, the optimizing control vectors are determined by analytical expressions.

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**The method is exact except for small errors involved in linearizing the equations of motion.

THE VARIATION OF PARAMETER EQUATIONS

The Equations of Perturbed Motion

The motion of a celestial object is approximately defined by a Keplerian orbit which assumes that the only force on the object is due to the force field of the central body. A more precise determination of the motion of the object requires taking into account the perturbing effects of the gravitational fields of other celestial bodies. In the case of satellites any nonsphericity of the central body will also perturb the motion.

The integration of the equations of motion of a celestial object, when perturbing effects are included, is generally carried out by Cowell's method, Encke's method, or the variation of parameters* method.(6),(7),(8)

In what follows, the variation of parameters method will be used to determine the equations of perturbed motion.

The motion of a particle in an inverse square central force field, subject to a perturbing acceleration, may be described by the following set of second-order differential equations:

$$\frac{d^2x}{d\tau^2} + \frac{\mu x}{r^3} = a_x$$

$$\frac{d^2y}{d\tau^2} + \frac{\mu y}{r^3} = a_y$$

$$\frac{d^2z}{d\tau^2} + \frac{\mu z}{r^3} = a_z$$

where: $\mu = \left(\frac{k}{k'}\right)^2 (m_o + m_1)$

*Also referred to as the variation of elements and variation of arbitrary constants method.

$$\tau = k' (t - t_o)^*$$

a_x, a_y, and a_z = components of perturbing force per unit mass.

These three second-order equations may be transformed into a system of six first order equations[9]

$$\dot{x}_i = g_i (x_j;\tau) + f_i (x_j;\tau) , \qquad i, j = 1,\ldots,6$$

$$x_i(0) = c_i, \quad \dot{x}_i = \frac{dx_i}{d\tau}$$

where

$$\begin{array}{llllll}
x_1 = x & g_1 = x_2 & f_1 = 0 \\
x_2 = \dot{x} & g_2 = -\dfrac{\mu x_1}{r^3} & f_2 = a_x \\
x_3 = y & g_3 = x_4 & f_3 = 0 \\
x_4 = \dot{y} & g_4 = -\dfrac{\mu x_3}{r^3} & f_4 = a_y \\
x_5 = z & g_5 = x_6 & f_5 = 0 \\
x_6 = \dot{z} & g_6 = -\dfrac{\mu x_5}{r^3} & f_6 = a_z
\end{array}$$

The functions $g_i (x_j;\tau)$ contain the effect of the central force field on the motion of the particle, while the functions $f_i(x_j;\tau)$ represent the effect of the perturbing force. The c_i are arbitrary constants of the solution representing the initial values of the position and velocity components at $t = t_o$ or $\tau = 0$. When there is no perturbing force the system of first order equations reduces to

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*The proper choice of k' in this transformation results in a more convenient unit of time.

$$\frac{dx_i}{d\tau} = g_i\ (x_j;\tau) \qquad i,\ j = 1,\ldots,6$$

where

$$x_i(0) = c_i = x_i\ (p_1,\ldots,p_6;\ 0)$$

The parameters $p_1,\ldots,p_6$ are a set of orbital parameters defining the two-body orbit. The solution of the two-body equations is given by

$$x_i(\tau) = x_i\ (p_1,\ldots,p_6;\ \tau)\ , \quad i = 1,\ldots,6$$

The effect of a small perturbing force is to cause the orbital parameters $p_1,\ldots,p_6$ to vary slowly with time. Thus, $p_1,\ldots,p_6$ may be considered as a new set of variables, and the solution equations may be considered as a set of transformation equations from the variables $p_1,\ldots,p_6$ to $x_1,\ldots,x_6$.

If we take the total derivatives of the transformation equations with respect to time we obtain

$$\frac{dx_i}{d\tau} = \frac{\partial x_i}{\partial\tau} + \sum_{j=1}^{6} \frac{\partial x_i}{\partial p_j}\frac{dp_j}{d\tau} \qquad i = 1,\ldots,6$$

Substituting in the equations of motion we obtain

$$\frac{\partial x_i}{\partial\tau} + \sum_{j=1}^{6} \frac{\partial x_i}{\partial p_j}\ \frac{dp_j}{d\tau} = g_i\ (x_j;\tau) + f_i\ (x_j;\tau) \qquad i = 1,\ldots,6$$

The partial derivatives $\frac{\partial x_i}{\partial\tau}$ with $p_1,\ldots,p_6$ held constant are identical with $\frac{dx_i}{d\tau}$ in the equations for unperturbed motion. Thus

$$\frac{\partial x_i}{\partial\tau} = g_i(x_j;\tau) \qquad i = 1,\ldots,6$$

and the equations of perturbed motion become

$$\frac{\partial x_1}{\partial p_1}\frac{dp_1}{d\tau} + \ldots + \frac{\partial x_1}{\partial p_6}\frac{dp_6}{d\tau} = 0$$

$$\frac{\partial x_2}{\partial p_1}\frac{dp_1}{d\tau} + \ldots + \frac{\partial x_2}{\partial p_6}\frac{dp_6}{d\tau} = f_2\,(x_j;\tau)$$

$$\cdot \qquad \cdot \qquad \cdot \qquad \cdot \qquad \cdot$$

$$\frac{\partial x_6}{\partial p_1}\frac{dp_1}{d\tau} + \ldots + \frac{\partial x_6}{\partial p_6}\frac{dp_6}{d\tau} = f_6\,(x_j;\tau)$$

In matrix form these equations become

$$J\,\frac{dp}{d\tau} = f$$

or

$$\frac{dp}{d\tau} = J^{-1} f = Af$$

where

$$p = \left[p_1,\ldots,p_6\right]^T$$

$$f = \left[0,\ f_2,\ 0,\ f_4,\ 0,\ f_6\right]^T$$

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \frac{\partial x_1}{\partial p_2} & \cdots & \frac{\partial x_1}{\partial p_6} \\ \frac{\partial x_2}{\partial p_1} & \frac{\partial x_2}{\partial p_2} & \cdots & \frac{\partial x_2}{\partial p_6} \\ \cdot & & & \\ & \cdot & & \\ & & & \cdot \\ \frac{\partial x_6}{\partial p_1} & \frac{\partial x_6}{\partial p_2} & \cdots & \frac{\partial x_6}{\partial p_6} \end{bmatrix}$$

The matrix J is the Jacobian matrix of the transformation and has an inverse[9]. The vector-matrix equation in expanded form becomes

$$\frac{dp_1}{d\tau} = \frac{J_{21} f_2 + J_{41} f_4 + J_{61} f_6}{|J|}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{dp_6}{d\tau} = \frac{J_{26} f_2 + J_{46} f_4 + J_{66} f_6}{|J|}$$

Where J_{rs} is the cofactor of the element in the r-th row and s-th column of $|J|$. If the functions $f_i(x_j;\tau)$ are numerically small relative to the functions $g_i(x_j;\tau)$, the right hand members of the differential equations in the variables $p_1,\ldots,p_6$ are numerically small relative to the right hand members of the equations in $x_1,\ldots,x_6$, and the parameters $p_1,\ldots,p_6$ will vary more slowly than the coordinates $x_1,\ldots,x_6$. Then domain of validity of the solution in $p_1,\ldots,p_6$ is greatly enlarged over that for $x_1,\ldots,x_6$.[9] In fact, if the interval of time over which the equations of perturbed motion are applied is sufficiently short, then the $p_1,\ldots,p_6$ may be assumed constant on the right hand side of the equation for $\frac{dp}{d\tau}$. The solution is then determined approximately by quadratures, and in vector-matrix form

$$p(\tau_1) = p(0) + \int_0^{\tau_1} A\, f\, d\tau$$

where

$$p(0) = \left[p_1(0),\ldots,p_6(0)\right]^T$$

Lagrange's Brackets

In a practical case where the orbital parameters $p_1,\ldots,p_6$ are specified and the equations are to be obtained in literal form it may be simpler to solve for the time derivatives of the parameters by making use of Lagrange's brackets. The virtue of this method lies in the fact that many of the elements of the matrix to be inverted vanish, and possibly the resulting matrix can be reduced to a lower dimension than matrix J before inversion. Let both sides of the vector matrix differential equation

$$J \frac{dp}{d\tau} = f$$

be premultiplied by the matrix $J^T E$ giving

$$J^T E \, J \, \frac{dp}{d\tau} = J^T E \, f$$

where J^T is the transpose of J and E is the product of three E-matrices which interchange the first and second, third and fourth, and fifth and sixth columns of J^T and multiply the odd columns of the resulting matrix by - 1.[10] The matrix E is given by

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

The resulting equation in expanded form may be written as

$$\begin{bmatrix} [p_1, p_1] & [p_1, p_2] & \cdots & [p_1, p_6] \\ [p_2, p_1] & [p_2, p_2] & \cdots & [p_2, p_6] \\ \cdot & & & \\ & \cdot & & \\ & & & \cdot \\ [p_6, p_1] & [p_6, p_2] & \cdots & [p_6, p_6] \end{bmatrix} \begin{bmatrix} \frac{dp_1}{d\tau} \\ \frac{dp_2}{d\tau} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dp_6}{d\tau} \end{bmatrix} = \begin{bmatrix} f_2 \frac{\partial x_1}{\partial p_1} + f_4 \frac{\partial x_3}{\partial p_1} + f_6 \frac{\partial x_5}{\partial p_1} \\ f_2 \frac{\partial x_1}{\partial p_2} + f_4 \frac{\partial x_3}{\partial p_2} + f_6 \frac{\partial x_5}{\partial p_2} \\ \\ \\ \\ f_2 \frac{\partial x_1}{\partial p_6} + f_4 \frac{\partial x_3}{\partial p_6} + f_6 \frac{\partial x_5}{\partial p_6} \end{bmatrix}$$

The elements of the matrix $J^T E J$ are known as Lagrange's brackets, and they possess the following properties:(11)

$$\left.\begin{aligned} [p_i, p_i] &= 0 \\ [p_i, p_j] &= -[p_j, p_i] \\ \frac{\partial [p_i, p_j]}{\partial t} &= 0 \end{aligned}\right\} \quad i, j = 1, \ldots, 6$$

where
$$[p_i, p_j] = \frac{\partial x_1}{\partial p_i}\frac{\partial x_2}{\partial p_j} - \frac{\partial x_2}{\partial p_i}\frac{\partial x_1}{\partial p_j} + \frac{\partial x_3}{\partial p_i}\frac{\partial x_4}{\partial p_j} - \frac{\partial x_4}{\partial p_i}\frac{\partial x_3}{\partial p_j} + \frac{\partial x_5}{\partial p_i}\frac{\partial x_6}{\partial p_j} - \frac{\partial x_6}{\partial p_i}\frac{\partial x_5}{\partial p_j}, \quad i, j = 1, \ldots, 6$$

The properties of Lagrange's brackets simplify the matrix $J^T E J$ in the following ways:

- o From the first property the elements on the main diagonal of $J^T E J$ vanish.
- o From the first two properties, the matrix $J^T E J$ is skew-symmetric.
- o From the third property the brackets do not contain time explicitly, and thus they may be evaluated for any epoch. The choice $t = t_o$ greatly simplified the expressions for the brackets.

The next step in the derivation involves obtaining the required partial derivatives, and from them evaluating the non-zero Lagrange's brackets.

Derivatives of the Disturbing Function

For the case of celestial bodies where the components of the perturbing force a_x, a_y, and a_z are the derivatives of a disturbing function R, i.e.,

$$a_x = \frac{\partial R}{\partial x} \qquad a_y = \frac{\partial R}{\partial y} \qquad a_z = \frac{\partial R}{\partial z}$$

The vector $J^T E f$ is given by

$$J^T E\, f = \begin{bmatrix} \frac{\partial R}{\partial p_1} \\ \frac{\partial R}{\partial p_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial R}{\partial p_6} \end{bmatrix}$$

The vector on the right follows from the definitions of f_2, f_4, f_6, x_1, x_3, and x_5 since

$$\frac{\partial R}{\partial x} \frac{\partial x}{\partial p_1} + \frac{\partial R}{\partial y} \frac{\partial y}{\partial p_1} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial p_1} = \frac{\partial R}{\partial p_1}$$

and

$$x(\tau) = x_1 (p_1, \ldots, p_6; \tau)$$

$$y(\tau) = x_3 (p_1, \ldots, p_6; \tau)$$

$$z(\tau) = x_5 (p_1, \ldots, p_6; \tau)$$

Inverting the matrix $J^T E\, J$ gives

$$\frac{dp}{d\tau} = (J^T E\, J)^{-1} J^T E\, f = A\, f$$

Because the first, third, and fifth components of vector f are zero, the matrix A may be rewritten as a 6 x 3 matrix and vector f may be written as a 3 x 1 vector. These revised forms of A and f will be assumed in all that follows.

A MULTISTAGE ORBIT CORRECTION PROCESS

The State Transformation Equation

A multistage orbit correction process is defined as consisting of a series of consecutive thrust vectors applied to vehicle so as to transfer it from one orbit or trajectory to some other. The process may be more precisely defined by specifying the direction and magnitude of the thrust vector for each separate stage of the process. Alternatively, the change in the acceleration vector of the vehicle produced by each thrust vector or the incremental velocity vector acquired by the vehicle during each stage of the process may be specified. It is assumed that the initial and final orbits have been predetermined by some measuring process.

In the discussion that follows the variation of parameter equations derived above are used to define the perturbed motion of the vehicle. The method of dynamic programming is used to select an optimum set of incremental velocity vectors defining the orbit transfer process. The choice of these incremental velocity vectors is based upon minimizing the sum of the squares of the weighted orbital parameter errors at the termination of the multistage process and is constrained by the total amount of propulsive energy available for the process. The equations derived are in such a form that certain of the final orbital parameter errors may be weighted more heavily than others if desired.

The variation of parameter equations for the perturbed motion in vector matrix form are given by

$$\frac{dp}{d\tau} = A\,f$$

where

$$p = \left[p_1, \cdots, p_6\right]^T$$

A = 6 x 3 matrix whose elements are functions of $p_1, \ldots, p_6$ and τ implicitly through the true, eccentric, or mean anomaly.

$$f = \left[f_1, f_2, f_3\right]^T$$

It was pointed out previously that if the interval of integration is sufficiently short and if the components of vector f are small enough, then the elements of matrix A may be assumed constant without making important errors. Although vector f may not be small relative to vector g, the duration of integration will be kept small enough relative to the orbital period so that the changes required in $p_1, \ldots, p_6$ will be small.

The acceleration vector f is due to the application of thrust by a rocket motor. If matrix A is assumed constant over the interval $0 \le \tau \le \tau_1$ then

$$\int_0^{\tau_1} A\, f\, d\tau = A \int_0^{\tau_1} f(\tau)\, d\tau = A\, \Delta V(\tau_1)$$

where

$$\int_0^{\tau_1} f(\tau)\, d\tau = \left[\int_0^{\tau_1} f_1(\tau)\, d\tau, \int_0^{\tau_1} f_2(\tau)\, d\tau, \int_0^{\tau} f_3(\tau)\, d\tau\right]^T$$

$$= \left[\Delta V_1(\tau_1)\,,\ \Delta V_2(\tau_1)\,,\ \Delta V_3(\tau_1)\right]^T$$

$$= \Delta V(\tau_1)$$

The vector ΔV is the incremental velocity vector acquired during the period of thrust from 0 to τ_1. The vector-matrix differential equation may then be integrated to

$$p(\tau_1) = p(0) + A\,\Delta V(\tau_1)$$

This equation represents the change in the state of the system as represented by the orbital parameters from an initial state $p(0)$ to some state $p(\tau_1)$ due to acquiring the vector velocity increment $\Delta V(\tau_1)$. The behavior of the acceleration vector f in producing ΔV is unimportant at this point (i.e., due to constant thrust, constant acceleration, etc.).

A somewhat simplified expression for the state transformation is given by

$$p_{k+1} = p_k + A_k\,\Delta V_k$$

where:

$p_k = p(\tau_k)$

$p_{k+1} = p(\tau_{k+1})$

A_k = Matrix A with its elements evaluated at $\tau = \tau_{k+1}$

ΔV_k = Incremental velocity vector acquired during the time interval from τ_k to τ_{k+1}

Consistent with the discussion of the variation of parameter equations the following assumptions are associated with the state transformation equation:

- The time interval $\tau_{k+1} - \tau_k$ is small relative to the orbital period for all values of $k = 0, 1, \ldots, N$
- The elements of the matrix A_k are taken as constants during the time interval $\tau_{k+1} - \tau_k$ and are evaluated at $\tau = \tau_{k+1}$.

The System Performance Index

There are two requirements placed on the performance of the system.

- The sum of the weighted squares of the errors remaining in the orbital parameters at the termination of the orbit correction process is to be a minimum.
- The total propulsive energy available to accomplish the orbit correction process is limited.

The performance index is designed to exhibit system behavior consistent with these two requirements.

Let p_N be the vector representing the desired set of orbital parameters at the termination of the orbit correction process. Then define

$$\delta_k = p_N - p_k$$

as the error remaining in the orbital parameter vector at $\tau = \tau_k$. Using the state transformation equation we may write

$$\delta_k = p_N - p_{k-1} - A_{k-1}\,\Delta V_{k-1}$$

or

$$\delta_N = p_N - p_{N-1} - A_{N-1}\,\Delta V_{N-1}$$

Here, δ_N represents the errors remaining in the orbital parameters at the termination of the orbit correction process. The sum of the weighted squares of the errors may be expressed as

$$\delta_N^T\, Q_N\, \delta_N = q_{11}(\delta p_1)^2 + q_{22}(\delta p_2)^2 + \ldots + q_{66}(\delta p_6)^2$$

where:

$$\delta_N = \left[\delta p_1,\ \delta p_2,\ \ldots,\ \delta p_6\right]^T$$

$\delta p_1, \ldots, \delta p_6$ = Errors remaining in orbital elements at the termination of the orbit correction process.

Q_N = 6 x 6 diagonal matrix with diagonal elements $q_{11}, \ldots, q_{66}$

$q_{11}, \ldots, q_{66}$ = Weighting factors for orbital parameter errors

The kinetic energy added to or subtracted from the vehicle during the orbit correction process depends upon the sum of the squares of the magnitudes of the incremental velocity vectors for all stages of the process.

$$\Delta V_k^T \Delta V_k = (\Delta V_k)_1^2 + (\Delta V_k)_2^2 + (\Delta V_k)_3^2$$

where $(\Delta V_k)_1$, $(\Delta V_k)_2$, $(\Delta V_k)_3$ are the components of ΔV_k.

The two requirements on system behavior combine to give the following performance index for an N-stage process beginning at state $\Delta p(0)$:

$$J_N \left[\Delta p(0);\ \Delta V_o, \ldots, \Delta V_{N-1}\right] = \delta_N^T Q_N \delta_N + \lambda \sum_{k=0}^{N-1} \Delta V_k^T \Delta V_k$$

where λ is a constant determined by available propulsive energy. This performance index is to be minimized by the proper choice of the N incremental velocity vectors $\Delta V_o, \ldots, \Delta V_{N-1}$.

Minimization of the Performance Index by the Method of Dynamic Programming

Minimization of J_N requires the proper choice of N vectors, i.e., it is an N-dimensional problem which would be rather lengthy to solve by conventional methods. The use of dynamic programming reduces the minimization problem to N one-dimensional problems which are much simpler to handle.[12],[13]

In what follows dynamic programming will be applied to J_N to derive the necessary recurrence relations to permit the computation J_N, δ_N, and ΔV_k, $k = 0, 1, \ldots, N-1$.

Consider first a one-stage process beginning at τ_{N-1} and terminating at τ_N. The performance index becomes

$$J_1 = \delta_N^T Q_N \delta_N + \lambda V_{N-1}^T V_{N-1}$$

where
$$\delta_N = \Delta p_{N-1} - A_{N-1} \Delta V_{N-1}$$

$$\Delta p_{N-1} = p_N - p_{N-1}$$

Substituting for δ_N in J_1 gives

$$J_1 = \left[\Delta p_{N-1} - A_{N-1}\,\Delta V_{N-1}\right]^T Q_N \left[\Delta p_{N-1} - A_{N-1}\,\Delta V_{N-1}\right] + \lambda \Delta V_{N-1}^T\,\Delta V_{N-1}$$

$$= \Delta\, p_{N-1}^T\, Q_N\, \Delta p_{N-1} - 2\,\Delta V_{N-1}^T\, A_{N-1}^T\, Q_N\, \Delta p_{N-1} + \Delta V_{N-1}^T\,(A_{N-1}^T\, Q_N\, A_{N-1} + \lambda\, I_3)\,\Delta V_{N-1}$$

where

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Differentiating J_1 with respect to ΔV_{N-1}, setting the result equal to zero, and solving for ΔV_{N-1} gives

$$\Delta V_{N-1} = (A_{N-1}^T\, Q_N\, A_{N-1} + \lambda\, I_3)^{-1}\;A_{N-1}^T\, Q_N\, \Delta p_{N-1}$$

$$= M_{N-1}\, A_{N-1}^T\; Q_N\, \Delta p_{N-1}$$

where

$$M_{N-1} = (A_{N-1}^T\, Q_N\, A_{N-1} + \lambda\, I_3)^{-1}$$

The above equation gives the vector ΔV_{N-1} which minimizes the one-stage performance index J_1 for the system initially in an arbitrary state p_{N-1}. Substituting this equation in J_1 gives the minimum value of J_1.

$$\text{Min } J_1\left[p_{N-1};\, \Delta V_{N-1}\right] = \Delta p_{N-1}^T\, Q_N\, \Delta p_{N-1} - 2\,\Delta p_{N-1}^T\, Q_N^T\, A_{N-1}\, M_{N-1}^T\, A_{N-1}^T\, Q_N\, \Delta p_{N-1}$$

$$+ \Delta p_{N-1}^T\, Q_N^T\, A_{N-1}\, M_{N-1}^T\, M_{N-1}^{-1} M_{N-1}\, A_{N-1}^T\, Q_N\, \Delta p_{N-1}$$

$$= \Delta p_{N-1}^T\,\left[Q_N - Q_N^T\, A_{N-1}\, M_{N-1}^T\, A_{N-1}^T\, Q_N\right]\,\Delta p_{N-1}$$

$$= \Delta p_{N-1}^T\, Q_{N-1}\, \Delta p_{N-1}$$

where

$$Q_{N-1} = Q_N - Q_N^T A_{N-1} M_{N-1}^T A_{N-1}^T Q_N$$

Also

$$\delta_N = \Delta p_{N-1} - A_{N-1} M_{N-1} A_{N-1}^T Q_N \Delta p_{N-1}$$

$$= (I_6 - A_{N-1} M_{N-1} A_{N-1}^T Q_N) \Delta p_{N-1}$$

Next consider a two-stage process beginning with the system in state p_{N-2} and terminating in state p_N. The one-stage process considered above becomes the second stage of the two-stage process, and the expression for J_2 is

$$J_2 \left[p_{N-2}; \Delta V_{N-1}, \Delta V_{N-2}\right] = \delta_N^T Q_N \delta_N + \lambda (\Delta V_{N-1}^T \Delta V_{N-1} + \Delta V_{N-2}^T \Delta V_{N-2})$$

$$= J_1 \left[p_{N-1}; \Delta V_{N-1}\right] + \lambda \Delta V_{N-2}^T \Delta V_{N-2}$$

At this point the principle of optimality is applied[13] which means the following: An optimal sequence of incremental velocity vectors ΔV_o, $\Delta V_1, \ldots, \Delta V_{N-1}$ has the property that whatever the initial p_o may be, and whatever choice is made for ΔV_o, the remaining sequence $\Delta V_1, \ldots, \Delta V_{N-1}$ must constitute an optimal sequence with regard to the state p_1 resulting from the choice of ΔV_o. This principle is applied by replacing $J_1 \left[p_{N-1}; \Delta V_{N-1}\right]$ by its minimized form $\Delta p_{N-1}^T Q_{N-1} \Delta p_{N-1}$ in the expression for J_2 and minimizing J_2 with respect to ΔV_{N-2}

$$\underset{\Delta V_{N-1}, \Delta V_{N-2}}{\text{Min}} J_2 \left[p_{N-2}; \Delta V_{N-1}, \Delta V_{N-2}\right] = \underset{\Delta V_{N-2}}{\text{Min}} \left(\Delta p_{N-1}^T Q_{N-1} \Delta p_{N-1} + \lambda \Delta V_{N-2}^T \Delta V_{N-2}\right)$$

where from the state transformation equation

$$\Delta p_{N-1} = p_N - p_{N-1}$$

$$= p_N - p_{N-2} - A_{N-2}\,\Delta V_{N-2}$$

$$= \Delta p_{N-2} - A_{N-2}\,\Delta V_{N-2}$$

The expression to be minimized with respect to ΔV_{N-2} becomes

$$\Delta p_{N-2}^T\, Q_{N-1}\, \Delta p_{N-2} - 2\, \Delta V_{N-2}^T\, A_{N-2}^T\, Q_{N-1}\, \Delta p_{N-2} + \Delta V_{N-2}^T\, (A_{N-2}^T\, Q_{N-1}\, A_{N-2} + \lambda I_3)\, \Delta V_{N-2}$$

Proceeding as for J_1 we obtain for the optimal ΔV_{N-2}

$$\Delta V_{N-2} = (A_{N-2}^T\, Q_{N-1}\, A_{N-2} + \lambda I_3)^{-1}\, A_{N-2}^T\, Q_{N-1}\, \Delta p_{N-2}$$

$$= M_{N-2}\, A_{N-2}^T\, Q_{N-1}\, \Delta p_{N-2}$$

where

$$M_{N-2} = (A_{N-2}^T\, Q_{N-1}\, A_{N-2} + \lambda I_3)^{-1}$$

The minimum value of J_2 becomes

$$\min_{\Delta V_{N-1}, \Delta V_{N-2}} J_2 \left[p_{N-2};\ \Delta V_{N-1},\ \Delta V_{N-2}\right] = \Delta p_{N-2}^T\, Q_{N-2}\, \Delta p_{N-2}$$

where

$$Q_{N-2} = Q_{N-1} - Q_{N-1}^T\, A_{N-2}\, M_{N-2}^T\, A_{N-2}^T\, Q_{N-1}$$

The expression for δ_N for the two-stage process is somewhat more complicated than for the one-stage process

$$\begin{aligned}
\delta_N &= \Delta p_{N-1} - A_{N-1}\, \Delta V_{N-1} \\
&= \left[I_6 - A_{N-1}\, M_{N-1}\, A_{N-1}^T\, Q_N\right] \quad \Delta p_{N-1} \\
&= \left[I_6 - A_{N-1}\, M_{N-1}\, A_{N-1}^T\, Q_N\right] \left[\Delta p_{N-2} - A_{N-2}\, \Delta V_{N-2}\right] \\
&= \left[I_6 - A_{N-1}\, M_{N-1}\, A_{N-1}^T\, Q_N\right] \left[I_6 - A_{N-2}\, M_{N-2}\, A_{N-2}^T\, Q_{N-1}\right] \Delta p_{N-2}
\end{aligned}$$

The above process may be carried on for N-stages. However, by carrying on for several more stages it is evident that recurrence relations exist for each of the desired quantities. These recurrence relations are

$$\left.\begin{aligned}
\Delta V_{N-r} &= M_{N-r}\, A_{N-r}^T\, Q_{N-r+1}\, \Delta p_{N-r} \\
M_{N-r} &= (A_{N-r}^T\, Q_{N-r+1}\, A_{N-r} + \lambda I_3)^{-1} \\
\text{Min } J_r &= \Delta p_{N-r}^T\, Q_{N-r}\, \Delta p_{N-r} \\
Q_{N-r} &= Q_{N-r+1} - Q_{N-r+1}^T\, A_{N-r}\, M_{N-r}^T\, A_{N-r}^T\, Q_{N-r+1} \\
\delta_N &= \prod_{r=1}^{N} (I_6 - A_{N-r}\, M_{N-r}\, A_{N-r}^T\, Q_{N-r+1})\, \Delta p_o
\end{aligned}\right\} \quad r = 1,\ldots,N$$

The evaluation of the matrix A_{N-r} for each time τ_{N-r+1} presents a problem. For a system described by a linear, constant coefficient differential equation the matrix A_{N-r} is constant. In the problem considered here the matrix A is a function of the components of the state vector p (i.e., the two-body orbital parameters) as well as time. However, as pointed out in Ref. (9), if the time interval over which the differential equations are integrated is sufficiently short, the components of vector p entering into the elements of A may be considered as constants for the entire process. This assumption is justified by the fact that the total orbital change as defined by $p_N - p_o$ will of necessity have to be small because of restrictions on available fuel.

An alternative procedure involves the following steps

- o Assume A constant for the entire process and compute the optimal sequence of ΔV_k, $k = 1, \ldots, N-1$.
- o Using the optimal ΔV_k compute p_k and then A_k.
- o Repeat process using A_k in place of constant A matrix and determine a new optimal sequence of ΔV_k.
- o Iterate the above procedure until the change in ΔV_k is sufficiently small.

The evaluation of the parameter λ is carried out by choosing several values of λ and computing the corresponding sequences of optimal ΔV_k. The resulting ΔV_k are tested in

$$\sum_{k=0}^{N-1} \Delta V_k^T \Delta V_k \leq C$$

For the maximum $p_N - p_o$ expected, the value of λ can be selected to satisfy the above inequality. The constant C on the right side of the inequality is proportional to the total kinetic energy change possible with the available fuel.

SOME SPECIAL CASES

Single Stage Orbit Correction Process

It is possible to transfer from the initial orbit to one close to the desired orbit by one application of thrust. How close the final orbit and the desired orbit are will depend upon the weights assigned to the orbital parameter errors between the initial and desired orbits and the position in the initial orbit at which the correction process is carried out. The possible advantage of a single stage orbit transfer process is that it may result in a final orbit satisfying practical requirements with less expenditure of fuel and with a less complex guidance system mechanization.

The variation of parameter equations derived in Section II are given by the vector-matrix equation

$$\frac{dp}{d\tau} = Af$$

which integrates to

$$p(\tau_1) = p(0) + \int_0^{\tau_1} Af \, d\tau$$

$$= p(0) + A \, \Delta V(\tau_1)$$

If we define $\Delta p(0)$ as the change required in the orbital parameter vector to transfer from the initial orbit to the desired orbit, then the residual errors in the orbital parameters at the end of the single stage orbit transfer process is given by

$$\delta = \Delta p(0) - A \, \Delta V(\tau_1)$$

where δ is the 6×1 residual orbital parameter error vector. It will be possible to choose $\Delta V(\tau_1)$ to make δ vanish only for certain special conditions.

Since these conditions will normally not be satisfied in practical cases it is necessary to choose the components of $\Delta V(\tau_1)$ so that δ will be minimized in accordance with some performance index. The one chosen for discussion here is to minimize the sum of the weighted squares of the components of δ, i.e., to minimize

$$J_1 = \delta^T Q \delta$$

where Q is a 6 x 6 diagonal matrix whose non-zero elements weight the squares of the individual orbital parameter errors. From the multi-stage process derivation the minimizing velocity vector is given by

$$\Delta V(0) = (A^T Q A)^{-1} A^T Q \Delta p(0)$$

since $\lambda = 0$ for the performance index considered.

The residual error vector δ is also given by

$$\delta = \left[I_6 - A(A^T Q A)^{-1} A^T Q\right] \Delta p(0)$$

$$= N \Delta p(0)$$

For δ to vanish the matrix N must be singular. It may be shown that this will be true only when

$$\Delta\vec{r} + \alpha \Delta\dot{\vec{r}} = 0 \text{ , } \Delta\vec{r} = \text{ error in position vector}$$

$$\alpha \approx \frac{1-\cos n\tau}{n \sin n\tau}$$

n = Mean daily motion

τ = Duration of thrust for single stage process.

The expression for α assumes that the magnitude of the thrust vector is so controlled that a constant acceleration due to the correcting thrust vector is maintained and the distance from the center of force is constant during the orbit correction process.

Two-Stage Orbit Correction Process

It is shown in Ref. (1) that two successive applications of thrust are required in general to completely remove the errors in the orbital parameters. There the system is analyzed in terms of rectangular components of position and velocity, and no constraint is placed on the amount of energy available to make the complete orbit correction. This situation may be treated as a special case of the multi-stage process if we assume a two stage process with $\delta_N^T \delta_N$ and λ equal to zero.

We have

$$\delta_N = p_N - p_{N-1} - A_{N-1}\,\Delta V_{N-1} = 0$$

$$p_{N-1} = p_{N-2} + A_{N-2}\,\Delta V_{N-2}$$

or

$$p_N - p_{N-2} - A_{N-1}\,\Delta V_{N-1} - A_{N-2}\,\Delta V_{N-2} = 0$$

This equation may be written as

$$A_{N-1}\,\Delta V_{N-1} + A_{N-2}\,\Delta V_{N-2} = \Delta p_{N-2}$$

This vector matrix equation represents six linear equations in the six unknown components of ΔV_{N-1} and ΔV_{N-2}. Let ΔV_{N-1} and ΔV_{N-2} make up a 6 x 1 vector,

$$\left[\Delta V_{N-1} \vdots\, \Delta V_{N-2}\right]^T = \left[(\Delta V_{N-1})_1, (\Delta V_{N-1})_2, (\Delta V_{N-1})_3, (\Delta V_{N-2})_1, (\Delta V_{N-2})_2, (\Delta V_{N-2})_3\right]^T$$

and the elements of A_{N-1} be designated by α_{jk}, and the elements of A_{N-2} by β_{jk}. We may write

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \beta_{11} & \beta_{12} & \beta_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \beta_{21} & \beta_{22} & \beta_{23} \\ \cdot & & & & & \\ & & \cdot & & & \\ & & & & \cdot & \\ \alpha_{61} & \alpha_{62} & \cdot & \cdot & \cdot & \beta_{63} \end{bmatrix} \begin{bmatrix} (\Delta V_{N-1})_1 \\ (\Delta V_{N-1})_2 \\ (\Delta V_{N-1})_3 \\ (\Delta V_{N-2})_1 \\ (\Delta V_{N-2})_2 \\ (\Delta V_{N-2})_3 \end{bmatrix} = \begin{bmatrix} (\Delta p_{N-2})_1 \\ (\Delta p_{N-2})_2 \\ (\Delta p_{N-2})_3 \\ \cdot \\ \cdot \\ \cdot \\ (\Delta p_{N-2})_6 \end{bmatrix}$$

If the time between application of the two thrust vectors is short then the first and fourth, second and fifth, and third and sixth columns of the $\alpha\ \beta$ matrix will be almost equal. This means that the determinant of the $\alpha\ \beta$ matrix will be quite small, resulting in very large components of velocity for a given parameter error vector.

A Special Performance Index

Consider the case of a communication satellite in 24-hour, circular equatorial orbit. Due to perturbing accelerations such a satellite will wander from its desired reference point and eventually move out of the ground antenna cone of view (assuming a non-steerable antenna). The problem is to periodically apply thrust to the satellite so as to keep it within the antenna's cone of view.

From the standard equation for a cone of revolution we may derive the equation

$$\tan^2 \phi = \frac{(\Delta A \cos h)^2 + (\Delta h)^2}{(1 + \frac{\Delta\rho}{\rho})^2}$$

$$\approx \quad (\Delta A \cos h)^2 + (\Delta h)^2$$

where ϕ = The cone's generating angle

ΔA = The angular azimuth error of the satellite position at the ground antenna.

Δh = The angular elevation error of the satellite position at the ground antenna.

If $\Delta A \cos h$ and Δh are expressed in terms of the errors in the six parameters defining the satellite's orbit $\tan^2 \phi$ becomes a quadratic form in the orbital parameter errors Δp_i, $i = 1,\ldots,6$.

$$\tan^2 \phi = \Delta p^T Q \, \Delta p$$

where $\Delta p = \left[\Delta p_1, \ldots, \Delta p_6\right]^T$

Q = 6 x 6 matrix whose elements are functions of local sidereal time and the position of the satellite in its orbit.

At any instant of time the satellite may be assumed to lie on the surface of a right circular cone whose vertex is at the ground antenna and whose axis is the line of sight from the antenna to the preferred satellite position in a perfect 24-hour, circular, equatorial orbit. If ϕ_r is the generating angle of the antenna cone of view we desire to maintain the following inequality.

$$\tan^2 \phi_r \geq \Delta p^T Q \, \Delta p$$

At any instant then, the angle ϕ defined by

$$\phi = \tan^{-1} \left[\Delta p^T Q \, \Delta p\right]^{\frac{1}{2}}$$

is a measure of system performance. Since minimizing ϕ is the same as minimizing $\tan^2 \phi$ the performance index for the system is

$$J = \tan^2 \phi = \delta^T Q \delta$$

where δ is a 6 x 1 vector whose components are the errors in the orbital parameters remaining at the termination of the orbital transfer process.

The performance index $\tan^2 \phi$ may be minimized by dynamic programming techniques as indicated in Section III, and an energy constant may be included if desired. The matrix Q_N becomes the matrix Q in the above equation and now has time dependent non-zero off diagonal elements. However, since the orbit correction process is normally of short duration relative to the orbital period, the elements of Q may be evaluated for the time of the termination of the process and assumed constant during the process.

Further Considerations

The general multi-stage orbit transfer process represents an N-dimensional minimization problem. A solution by classical minimization techniques requires solving N simultaneous equations. The use of dynamic programming can reduce the N-dimensional problem to N one-dimensional problems. Further, when the state transformation equation is linear and the system performance index is quadratic, an analytic solution can be obtained which includes certain types of constraints. The method presented in this paper is very flexible since

- Any constraint that can be related to a quadratic function of velocity may be incorporated.
- Any desired set of two-body orbital parameters may be used to formulate the equations of motion.
- By properly choosing the elements of matrix Q_N, the squares of the errors on the terminal orbital elements may be given any desired weight.

- In the multi-stage process the spacing of the corrective thrust vectors may be chosen to suit the particular problem at hand, i.e., rendezvous problem, point-to-point trajectory, satellite orbit, etc.

The recurrence relations were derived assuming a three-dimensional control vector ΔV. Using this formulation, as was pointed out in the discussion of the single-stage correction process, it is impossible to drive the terminal errors to zero, even with no constraints, except under special conditions. However, when one or two of the orbital parameter errors are of special significance they may be weighted very heavily and can be driven to very small values at the termination of the process. An alternative formulation is to use the two-stage process discussed previously as one stage of the multi-stage process. The $\alpha\ \beta$ matrix in the two-stage formulation becomes the A_{N-r} matrix in the recurrence relations, and the ΔV_{N-r} vector is now a six dimensional vector as defined for the two-stage process. Of course, for $\lambda = 0$ (no constraint) the multi-stage process degenerates to the two-stage process with zero terminal errors, i.e., $\delta_N = 0$.

A word of caution is in order concerning the use of the method for numerical computations. The variation of parameter equations are based on strong assumptions of linearity. Care must be taken in designing the process so that the resulting changes in orbital parameters during any one stage does not exceed the acceptable linear region of the state transformation equations.

ACKNOWLEDGMENT

This investimation was supported by the National Aeronautics and Space Administration, Contract-NASr-21(02).

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